

Wakefields and Spacecharge in ESME

- Core code
- Space charge model
- Longitudinal impedances and wakefields
- Frequency domain facilities
- Time domain facilities

I want to get here at least.

- Contributions from analytical dynamics
 - time or (equivalently) bandwidth scaling
 - symplecticity
 - micro-matching of initial distribution
- Transparency, accessibility, utility

Core Code

Multiparticle tracking has established utility for modeling evolution of longitudinal phase space distributions of particles in accelerators as they respond to the rf in acceleration or bunch manipulation. One goes from single particle to multiparticle dynamics by calculating the beam current every time step and including its effect on the single particle motion. However, the number of macroparticles needed and bandwidth required for quantities in the frequency domain need careful attention. It is very easy to generate spectacular spurious instabilities by long time steps or too few macroparticles.

The kernel of the code is the map for the single-particle EOM:

$$\varphi_{i,n} = \frac{\tau_{s,n-1}}{\tau_{s,n}} \varphi_{i,n-1} + 2\pi h(S_{i,n} - 1) \quad (1)$$

$$\varepsilon_{i,n} = \varepsilon_{i,n-1} + eV(\varphi_{i,n} + \phi_{s,n}) - eV(\phi_{s,n}) \quad (2)$$

Symbols Defined

ϕ	rf phase
ϕ_s	synchronous phase
φ_i	difference between particle phase and ϕ_s
i	index for particles
n	index for turns
h	rf harmonic number
e	elementary particle charge (> 0)
E_s	synchronous energy
ε_i	energy of i -th particle minus E_s
β	relativistic velocity v/c
γ	relativistic energy E_s/m_0c^2
γ_T	γ of transition energy in synchrotron
η	phase slip factor ($\gamma_T^{-2} - \gamma^{-2}$)
V	total potential
Z	longitudinal impedance
$\Omega_{s,n}$	angular frequency of synchronous particle circulation
$\Omega_{i,n}$	angular frequency of i -th particle circulation
$S_{i,n}$	per-turn slip $\Omega_{s,n}/\Omega_{i,n}$
$\omega_{s,n}$	small amplitude oscillation angular frequency
N	number of particles in beam
Ψ	particle density in phase space
Θ	azimuthal angle measured $\pm 180^\circ$ from rf
$\Lambda(\Theta)$	azimuthal charge density
λ	dimensionless real scaling constant

Space Charge Model

The self-impedance from the direct interparticle forces is derived from an electrostatic calculation in the beam rest frame transformed to the lab frame.[1] The force arises from the gradient of the azimuthal charge distribution $\Lambda_n(\Theta)$, which can be evaluated from the fourier series for Λ for the frequency domain or directly for time domain. The impedance representing the direct particle-particle force is

$$\frac{Z_m^{sc}}{m} = -i \frac{Z_0 g}{2\beta\gamma^2}, \quad (3)$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\Omega$ and g is a factor containing the dependence on beam and vacuum chamber transverse dimensions. For a uniform cylindrical beam of radius a centered in a smooth beampipe of radius b

$$g = 1 + 2 \log\left(\frac{b}{a}\right). \quad (4)$$

In ESME this factor is scaled with beam momentum to account for the change in beam radius and rolled off at very high frequency to approximate the exact solution at high mode numbers. The impedance is pure imaginary, like an energy-dependent negative inductance; the bunch as a whole can not gain or lose energy through internal forces.

Frequency Domain Facilities

ESME is fundamentally time domain, but beam current and image currents can be fourier analyzed at harmonics of synchronous circulation frequency $\Omega_{s,n}$. At the end of the n -th turn the beam has a distribution of particles in azimuth and total energy $\Psi_n(E, \Theta)$ normalized to N , the total number of particles in the beam. For small spread in particle velocity, the distribution circulates some few turns without change of form. Then the current distribution is just

$$I_{b,n}(\Theta) = \frac{\Omega_{n,s}e}{2\pi} \int \Psi_n(E, \Theta) dE = \frac{\Omega_{n,s}e}{2\pi} \Lambda_n(\Theta) . \quad (5)$$

The distribution $\Lambda_n(\Theta)$ circulates with negligible change of shape if

$$\frac{\sigma_\tau}{\tau} = \frac{\eta\sigma_E}{\beta^2 E_s} \ll 1 , \quad (6)$$

where the σ 's are the rms spreads resulting from the energy variance

$$\sigma_E^2 = \iint (E - E_s)^2 \Psi(E, \Theta) dE d\Theta . \quad (7)$$

Generally the width of the energy distribution will be severely limited by available aperture.

The mapping approach additionally requires what will be called a quasistatic current distribution $I_{b,n+1}(\Theta) \approx I_{b,n}(\Theta)$ implying $\Omega_{s,n} \approx \Omega_{s,n-1}$. If one wants to write the current at the gap as a function of time there is the usual change of sign between phase and azimuth:

$$I_b(\Omega_{st}) = \frac{e\Omega_s}{2\pi} \sum_m \Lambda_m e^{i(-m\Omega_{st} + \psi_m)} . \quad (8)$$

The fourier series for the current has been written with real amplitudes multiplied by complex phase factors, that is, as a sum of phasors.

The current produces a beam-induced voltage through the total longitudinal coupling impedance $Z_{||}(\omega)$; this quantity evaluated at $m\Omega_s$ will be denoted as the phasor $Z_m e^{i\chi_m}$. The beam-induced voltage is applied to each particle at time t_n when the synchronous particle is at the gap; that voltage depends on the relative phase between the particle and the current. The synchronous particle has phase 0. Thus, the energy increment for the i -th particle on the n -th turn resulting from the voltage induced by the beam current is

$$eV_{i,n}^b = -\frac{Ne^2\Omega_s}{2\pi} \sum_m \Lambda_{m,n} Z_m e^{i(m\Theta_{i,n} + \psi_{m,n} + \chi_m)}. \quad (9)$$

This sum involves phasors with time factors only multiples of $\Omega_{s,n}$; no synchrotron sidebands are evident. The same sum derived rigorously with a specialization to small amplitude oscillations is[2]

$$V_{\text{coll}}(\phi) = \frac{e\omega_0}{2\pi} \sum_{k=1}^N \sum_{p=-\infty}^{\infty} e^{ip\phi/h} \left\{ Z(p\omega_0) - \frac{i\omega_0 \hat{\tau}_k}{2} [(p + Q_s) Z(p\omega_0 + \omega_s) e^{i(\psi_k + Q_s\phi/h)} + (p - Q_s) Z(p\omega_0 - \omega_s) e^{-i(\psi_k + Q_s\phi/h)}] \right\}, \quad (10)$$

where the sum over k is taken over all particles. An upper and lower sideband are evident here; the whole spectrum of sidebands is presumably present in the time domain treatment with the nonlinear potential. It is a matter of long observation that the quasi-static frequency domain treatment rarely differs practically from the analogous time domain solution.

Time Domain Facilities

1. Gradient of Λ calculated at each particle by cubic spline interpolation of over bin of particle and two immediate neighbors
2. Green's function solution for simple resonance(s) with turn-to-turn accumulation
3. Wakefield for arbitrary distribution and arbitrary Z_{\parallel} using response to triangular unit current pulse calculated by TCBI or like
4. Free inter-mixing of time domain and frequency domain
5. Transient fundamental beam loading with optional feedback and/or feedforward correction

Scaling Concept

It appears that the phase space motion can be accelerated by scaling the phase slip factor η and the potential up by the same factor, hereafter λ . The potential is not necessarily just that from the rf system; the collective potential enters identically.

- The particle distributions are identical when time t in the un-scaled calculation is compared to time t/λ in the scaled calculation.
- Obvious gain is a factor λ^{-1} in the computing time by speeding up the clock in the scaled calculation.
- Scaling up of the time means that frequencies like the rf frequency and resonance frequencies in $Z_{||}$ must also be scaled.

- With broadband impedance or direct interparticle forces consequences of frequency scaling are different and very advantageous.
 - Impedance must be covered over some particular frequency bandwidth
 - Fourier components of the beam current must span bandwidth.
 - The charge is divided into $2W/f_0$ bins for the finite transform.
 - In the scaled system f_0 is λ times higher; the number of bins can be reduced by a factor λ^{-1} .
 - Sensitivity lost to features on the scale of f_0 in the frequency dependence of $Z_{||}$.
 - Major gain from scaling comes from possibility to reduce number of macro particles — λ^{-3} .

- Space charge term in frequency domain is

$$\frac{Z_{sc}}{n} = -i \frac{Z_0 g}{2\beta\gamma^2}$$

- n is the harmonic number
 - $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\Omega$
 - $\gamma = E/m_0c^2$
 - g is the geometric factor for the beam tube.
- Evaluation not compromised by more widely spaced frequency sampling
 - Number of macroparticles can be scaled by λ^{-3} with same level of numerical noise
 - shown rigorously for space charge[3]
 - heuristic frequency domain argument for any smooth $Z_{||}(\omega)$ [4]
 - Thus gain from scaling λ^4 in most cases

Utility of Scaling

- Very effective (λ^4) with space charge and smoothly varying impedance
- Less effective (λ^1) with dominant narrow resonances
- Limitation on λ by maximum allowable spacing of circulation harmonics
- Limitation on λ by acceptable step size
 - $\lambda > 1$ increases effective step size
 - steps in t' must be small fraction of τ_S
 - steps in t' must be small fraction of bunch length and bunch height

Symplecticity

With the substitution

$$W_{i,n} = \varepsilon_{i,n}/\omega_{s,n} \quad (11)$$

into the eqs. 1 and 2 one arrives at the map **M**:

$$\varphi_{i,n} = \frac{\omega_{s,n}}{\omega_{s,n-1}} \varphi_{i,n-1} + 2\pi h (S_{i,n} - 1) \quad (12)$$

$$W_{i,n} = \frac{\omega_{s,n-1}}{\omega_{s,n}} W_{i,n-1} + \frac{e}{\omega_{s,n}} [V(\varphi_{i,n} + \phi_{s,n}) - V(\phi_{s,n})] . \quad (13)$$

The Jacobian is

$$J(\mathbf{M}) = \frac{\partial(\varphi_{i,n}, W_{i,n})}{\partial(\varphi_{i,n-1}, W_{i,n-1})} \equiv 1 \quad (14)$$

$$= \left| \begin{array}{cc} \frac{\omega_{s,n}}{\omega_{s,n-1}} & \frac{e}{\omega_{s,n}} eV' \\ 2\pi h \frac{\partial S_{i,n}}{\partial W_{i,n-1}} & \frac{\omega_{s,n-1}}{\omega_{s,n}} + \frac{e}{\omega_{s,n}} V' 2\pi h \frac{\partial S_{i,n}}{\partial W_{i,n-1}} \end{array} \right|$$

Numerical Evaluation of Jacobian

The Jacobian can be evaluated directly from the numerical results of a single iteration of the map to establish that area is conserved and what is the effect of numerical precision. The next figure shows $1 - J$ evaluated with a differences of 1° and 1 MeV between macroparticles. It shows that the error is systematic, like $\cos \varphi$, with maximum magnitude $1 \cdot 10^{-10}$. The second figure shows the same quantity evaluated with a phase space cell of $.02^\circ \times .02$ MeV, which gives the minimum maximum magnitude of $3 \cdot 10^{-12}$ determined by offsetting of improved accuracy in the derivatives by increased truncation error with smaller spacing. The systematic form of $1 - J$ is still apparent but now partially obscured by numerical noise.

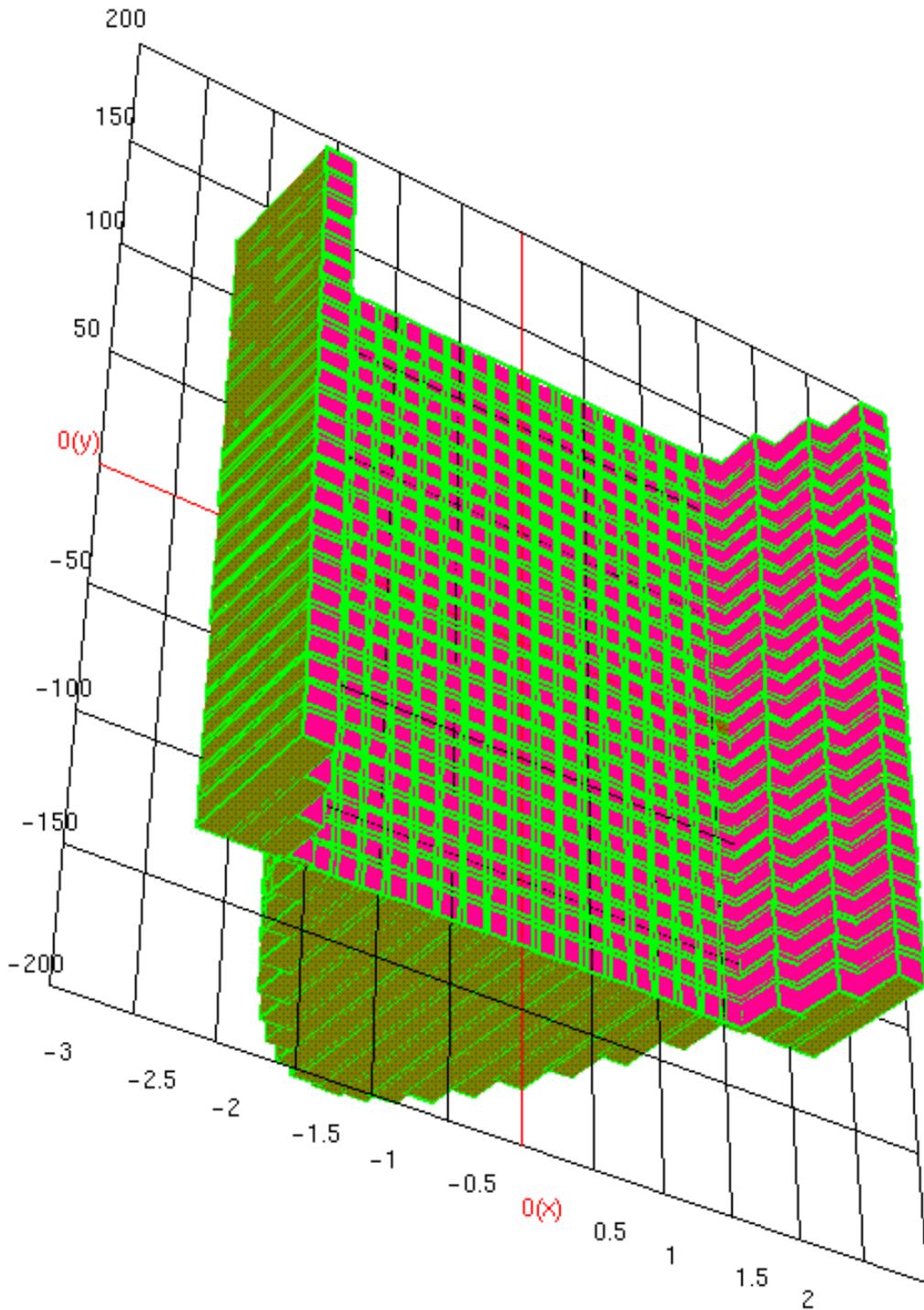


Fig. 1: Plot of $1 - J$ for synchrotron motion map on a grid with cells $0.2^\circ \times 0.2$ MeV, the parameters for a non-accelerating bucket at 150 GeV in the Tevatron. Max = $1 \cdot 10^{-10}$.

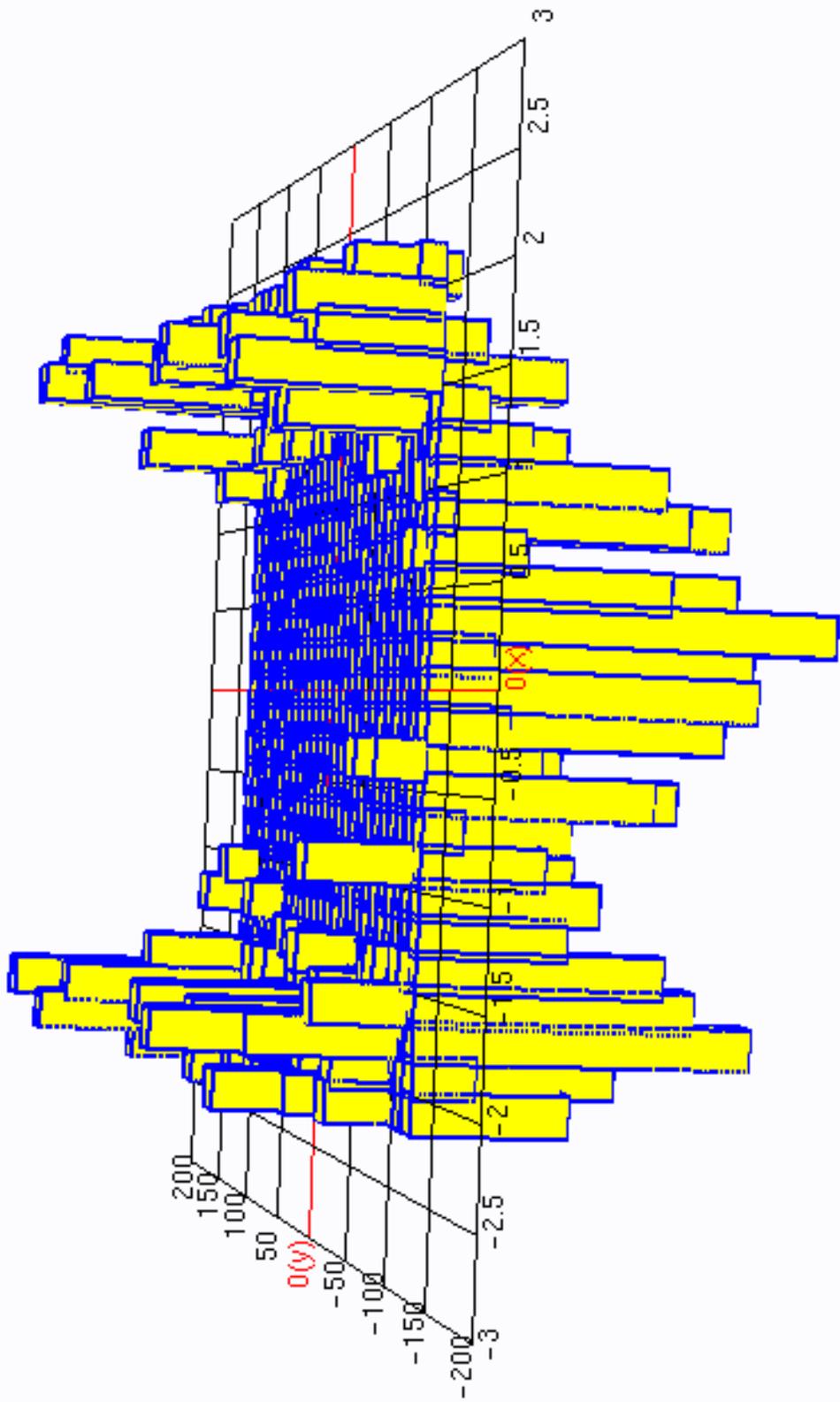


Fig. 2: Like Fig. 1 with cell size $0.02^\circ \times 0.02$ MeV.
Max = $3 \cdot 10^{-12}$.

Micro-matched Initial Distributions

- Tracking often shows charge-dependent emittance growth well below expected instability limits because initial distribution is not an equilibrium solution to EOM — not matched
- Even for single particle potentials there are only a few easily used well-matched initial distributions.
- The most general approach to obtaining a matched distribution is to transform adiabatically from a matched case to the desired case, e.g., by turning on beam charge slowly or by capturing slowly from a coasting beam.
- The elliptical distribution is matched at zero charge; it makes a reasonable starting point.

Some Practical Advantages

transparency There is a reasonably comprehensive and current user's guide. At least for simple applications, required input is modest and simply related to the ring being modeled.

accessibility Code is nearly standard f77, Linux compatible. Source and compiled versions are available via internet. Graphics package is fortran freeware.

utility Many features of practical machines have been introduced over years as part of Antiproton Source development, MI studies, Proton Driver, *et al.* Also requests for heavy ion provisions, special distributions, and so on have come from other Labs. The program architecture has absorbed most of this without grotesque elaboration, although there has been a proliferation of options and parameters that is probably distracting to a beginning user. (Remember what happened to word processing when people discovered Fonts?)

References

1. J. A. MacLachlan, “Longitudinal Phasespace Tracking with Spacecharge and Wall Coupling Impedance”, Fermilab FN-446 (2/87)
2. J. A. MacLachlan and Z. Nazario, “Scaling for Faster Macroparticle Simulation of Longitudinal Dynamics in Synchrotrons and Storage Rings”, PRST-AB **4** 019001 (2001)
3. J. Wei, “Longitudinal Dynamics of the Non-Adiabatic Regime on Alternating Gradient Synchrotrons”, PhD dissertation for State University of New York at Stony Brook (1990)
4. J. A. MacLachlan, “Particle Tracking in E - φ Space for Synchrotron Design and Diagnosis”, FERMILAB-Conf-92/333 (1992)
5. J. MacLachlan, “Difference Equations for Longitudinal Motion in a Synchrotron”, Fermilab note FN-529 (1989)
6. Etienne Forest, “Canonical Integrators as Tracking Codes (or How to Integrate Perturbation Theory with Tracking)”, SSC-138 (September 1987), unpublished

7. The current documentation, containing references to the underlying principles, is most accessible on the ESME web page www-ap.fnal.gov/ESME