Optimal Tune Measurement

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Tune Measurement Algorithm

According to W. March, the application pgm that computes Booster tunes currently works as follows:

- acquire N turns of BPM data x_i
- subtract $\langle x_i \rangle$ obtained by averaging over a sliding window (default: 10 samples)
- Take FFT $X = FFT(x - \langle x \rangle)$
- display power spectra $X^*conj(X)$ (or $|X|$) . Tune is maximum value.

Accuracy of the FFT

The error associated with the bare FFT algorithm in resolving the tune (a specific frequency peak) is

$$\varepsilon = 1/2N$$

For dynamic tune (chromaticity) measurements, we typically use 128 samples. The resolution is therefore ~ 0.01.

Can we Do Better with the Same no of Samples ?

- At first glance, the answer seems to be "no" because of that would violate the sampling theorem.

- However, the sampling theorem assumes that we have **no** information about the signal in-between samples. This is not true; we actually do ! In the neighborhood of a tune peak, we know that the signal is almost a pure sinusoidal (betatron oscillation). We can use this information to **interpolate** the spectrum.

- We also know that the signal measuring interval is not an exact multiple of the betatron period. Since the FFT implicitly assumes that the signal is the periodic extension of the measurement data, spurious frequencies are introduced into the spectrum. These frequencies can be eliminated using a window filter which effectively eliminates discontinuities at the boundaries of the FFT implicit period.

Hann(ing) Window

- Hann window function $\chi_\lambda = A_\lambda \, sin^\lambda(\pi n / N)$
- Premultiply data by χ(n) to eliminate "edge" discontinuities.
- Equivalent to a convolution of the spectrun with a sharply peaked function witout "sidelobes".
- Disadvantage: **broadens** the tune peak (but reduces error on the position of the maximum) . l>2 not desirable; (affect accuracy of interpolation) l=2 good compromise.
  Tune Error with Hann window: ε = $O(1/N^{(l+2)})$

Interpolation

- Interpolate in the neighborhood of the peak using the known continuous spectrum of a truncated sine. The "exact" tune is a free parameter.

$$\phi(j) = \left| \frac{\sin(N\pi(\nu^* - \nu_j))}{N \sin(\pi(\nu^* - \nu_j))} \right|$$

- Constraints: the interpolation function must coincide with the peak data point and the next highest data point.
- The constraints determine the "exact" tune ν^*.
- Tune Error: $\varepsilon = O(1/N^2)$

Even Better: Windowing + Interpolation

- In theory, tune error can be as good as $O(1/N^4)$!
- In practice there is noise; however, the combination windowing+interpolation always does much better than a simple FFT (at least $\sim O(1/N^2)$)
- This is easy to implement, so we should try !

References

- The problem of accurate and efficient determination of the tune arose in the context of the so-called frequency analysis technique pioneered by J. Laskar (a technique used in celestial mechanics and beam dynamics to locate boundaries of chaotic regions). The idea of interpolating the spectrum is often associated with Laskar, although it has also been used in industry.
- The Hann(ing) windowing technique is well-known and described in details in the standard specialized litterature.

The book "Numerical Recipes" offers a good discussion.
- "Algorithms for a Precise Determination of the Betatron Tune" R. Bartolinni et al. , EPAC 96 is a recent paper that summarizes the application of windowing and spectrum interpolation in the context of improving the accuracy of betatron tune measurements.